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Polar supermultiplets, hermitian symmetric spaces and hyperkähler metrics

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Abstract: We address the construction of four-dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models on tangent bundles of *arbitrary* Hermitian symmetric spaces starting from projective superspace. Using a systematic way of solving the (infinite number of) auxiliary field equations along with the requirement of supersymmetry, we are able to derive a closed form for the Lagrangian on the tangent bundle and to dualize it to give the hyperkähler potential on the cotangent bundle. As an application, the case of the exceptional symmetric space $E_6/\,\mathrm{SO}(10) \times \mathrm{U}(1)$ is explicitly worked out for the first time.

Keywords: Sigma Models, Extended Supersymmetry, Differential and Algebraic Geometry.

Contents

1. Introduction

The intimate relation between the number of supersymmetries and the target space geometry for supersymmetric sigma model [1] has been fruitfully exploited over the years. Here we are interested in the four-dimensional $\mathcal{N} = 2$ models whose target space is hyperkähler [2].

There are two methods for constructing new models from old ones; the Legendre transform and the hyperkähler reduction [3, 4], both of which have been reformulated in the manifest $\mathcal{N} = 2$ supersymmetric setting of projective superspace.

Projective superspace extends superspace at each point by an additional bosonic coordinate ζ which is a projective coordinate on $\mathbb{C}P^1$; actions are written using contour integrals over ζ, and reality conditions are imposed using complex conjugation of ζ composed with the antipodal map [5−8].

In a recent paper [9], we constructed, building in part on earlier work [10, 11], $\mathcal{N} = 2$ supersymmetric models on the tangent bundles of a large number of the Hermitian symmetric spaces as well as, using the generalized Legendre transform [6], the hyperkähler metrics on the corresponding cotangent bundles. Our approach rested on finding solutions to the $\mathcal{N} = 2$ projective superspace auxiliary field equations in Kähler normal coordinates at a point and then extending the solutions using cleverly chosen coset representatives. Although this method is perfectly viable, it becomes very cumbersome when more complicated spaces involving the exceptional groups are considered. For this reason we have changed the perspective in this paper. Our discussion is based on the solution to the auxiliary field equations originally described in [10, 11]. Starting from this solution and the duly modified second supersymmetry transformation allows us to completely determine the tangent-bundle action. We also describe how to find the dual cotangent-bundle action.

As illustrations of our method, we rederive some of the results in [9]. As a new application, we present a model on the tangent bundle of $E_6/\,\mathrm{SO}(10)\times\mathrm{U}(1)$ as well as the hyperkähler potential on the corresponding cotangent bundle.

The organization of the paper is as follows. In section two we describe the background material on $\mathcal{N}=2$ sigma models formulated using projective superspace. Our general construction is presented in section three. Section four contains the application to $E_6/\,\mathrm{SO}(10)\times\mathrm{U}(1)$, and in section five we give an alternative description of our Lagrangian, which leads to very direct relations to previous results but seems to have a more limited applicability. Examples are found in section five and in appendix A. Appendix B contains an explicit derivation of a relation used in section four.

2. Background material on $\mathcal{N}=2$ sigma models

We are interested in a family of 4D $\mathcal{N}=2$ off-shell supersymmetric nonlinear sigma-models that are described in ordinary $\mathcal{N}=1$ superspace by the action[1]

$$S[\Upsilon,\breve{\Upsilon}]=\frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}\zeta}{\zeta}\int\mathrm{d}^8z\,K\left(\Upsilon^I(\zeta),\breve{\Upsilon}^{\bar{J}}(\zeta)\right)\ . \tag{2.1}$$

The action is formulated in terms of the so-called *polar* multiplet [6, 7] (see also [8]), one of the most interesting $\mathcal{N}=2$ multiplets living in projective superspace. The polar multiplet is described by an arctic superfield $\Upsilon(\zeta)$ and antarctic superfield $\breve{\Upsilon}(\zeta)$ that are generated by an infinite set of ordinary $\mathcal{N}=1$ superfields:

$$\Upsilon(\zeta)=\sum_{n=0}^{\infty}\Upsilon_n\zeta^n=\Phi+\Sigma\,\zeta+O(\zeta^2)\,,\qquad\breve{\Upsilon}(\zeta)=\sum_{n=0}^{\infty}\bar{\Upsilon}_n(-\zeta)^{-n}\ . \tag{2.2}$$

Here Φ is chiral, Σ complex linear,

$$\bar{D}_{\dot\alpha}\Phi=0\,,\qquad\qquad\bar{D}^2\Sigma=0\,, \tag{2.3}$$

and the remaining component superfields are unconstrained complex superfields. The above theory occurs as a minimal $\mathcal{N}=2$ extension of the general four-dimensional $\mathcal{N}=1$ supersymmetric nonlinear sigma model [1]

$$S[\Phi,\bar{\Phi}]=\int\mathrm{d}^8z\,K(\Phi^I,\bar{\Phi}^{\bar{J}})\,, \tag{2.4}$$

with K the Kähler potential of a Kähler manifold $\mathcal{M}$.

The extended supersymmetric sigma model (2.1) inherits all the geometric features of its $\mathcal{N}=1$ predecessor (2.4). The Kähler invariance of the latter,

$$K(\Phi,\bar{\Phi})\quad\longrightarrow\quad K(\Phi,\bar{\Phi})\ +\ \Lambda(\Phi)+\bar{\Lambda}(\bar{\Phi}) \tag{2.5}$$

turns into

$$K(\Upsilon,\breve{\Upsilon})\quad\longrightarrow\quad K(\Upsilon,\breve{\Upsilon})\ +\ \Lambda(\Upsilon)\ +\ \bar{\Lambda}(\breve{\Upsilon}) \tag{2.6}$$

[1]The study of such models in this context was initiated in [12, 10, 11]. They correspond to a subclass of the general hypermultiplet theories in projective superspace [6, 7].

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for the model (2.1). A holomorphic reparametrization of the Kähler manifold,

$$\Phi^I \quad \longrightarrow \quad f^I\big(\Phi\big)\,, \tag{2.7}$$

has the following counterpart

$$\Upsilon^I(\zeta) \quad \longrightarrow \quad f^I\big(\Upsilon(\zeta)\big) \tag{2.8}$$

in the $\mathcal{N}=2$ case. Therefore, the physical superfields of the $\mathcal{N}=2$ theory

$$\Upsilon^I(\zeta)\Big|_{\zeta=0} ~=~ \Phi^I~, \qquad \frac{{\rm d}\Upsilon^I(\zeta)}{{\rm d}\zeta}\Big|_{\zeta=0} ~=~ \Sigma^I~, \tag{2.9}$$

should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point. Thus the variables (Φ^I, Σ^J) parametrize the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$ [12].

To describe the theory in terms of the physical superfields Φ and Σ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{{\rm d}\zeta}{\zeta}\,\zeta^n \,\frac{\partial K(\Upsilon, \breve{\Upsilon})}{\partial \Upsilon^I} ~=~ \oint \frac{{\rm d}\zeta}{\zeta}\,\zeta^{-n} \,\frac{\partial K(\Upsilon, \breve{\Upsilon})}{\partial \breve{\Upsilon}^{\bar I}} ~=~ 0~, \qquad n \geq 2~. \tag{2.10}$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar\Phi, \Sigma, \bar\Sigma)$ denote a unique solution subject to the initial conditions

$$\Upsilon_*(0) = \Phi~, \qquad \dot{\Upsilon}_*(0) = \Sigma~. \tag{2.11}$$

For a general Kähler manifold $\mathcal{M}$, the auxiliary superfields $\Upsilon_2, \Upsilon_3, \dots$, and their conjugates, can be eliminated only perturbatively. Their elimination can be carried out using the ansatz [13]

$$\Upsilon^I_n = \sum_{p=0}^{\infty} G^I{}_{J_1 \dots J_{n+p}\, \bar L_1 \dots \bar L_p}(\Phi, \bar\Phi)\, \Sigma^{J_1} \dots \Sigma^{J_{n+p}}\, \bar\Sigma^{\bar L_1} \dots \bar\Sigma^{\bar L_p}~, \qquad n \geq 2~. \tag{2.12}$$

Upon elimination of the auxiliary superfields, the action (2.1) takes the form [10, 11]

$$\begin{aligned} S_{\rm tb}[\Phi, \Sigma] ~&=~ \int {\rm d}^8 z\, \Big\{ K\big(\Phi, \bar\Phi\big) + \sum_{n=1}^{\infty} \mathcal{L}_{I_1 \cdots I_n \bar J_1 \cdots \bar J_n}\big(\Phi, \bar\Phi\big) \Sigma^{I_1} \dots \Sigma^{I_n} \bar\Sigma^{\bar J_1} \dots \bar\Sigma^{\bar J_n} ~\Big\} \\ &\equiv~ \int {\rm d}^8 z\, \Big\{ K\big(\Phi, \bar\Phi\big) + \sum_{n=1}^{\infty} \mathcal{L}^{(n)}\big(\Phi, \bar\Phi, \Sigma, \bar\Sigma\big) \Big\}~, \end{aligned} \tag{2.13}$$

where $\mathcal{L}_{I\bar J} ~=~ -g_{I\bar J}\big(\Phi, \bar\Phi\big)$ and the tensors $\mathcal{L}_{I_1 \cdots I_n \bar J_1 \cdots \bar J_n}$ for $n > 1$ are functions of the Riemann curvature $R_{I\bar J K \bar L}\big(\Phi, \bar\Phi\big)$ and its covariant derivatives. Each term in the action contains equal powers of Σ and $\bar\Sigma$, since the original model (2.1) is invariant under rigid U(1) transformations [10]

$$\Upsilon(\zeta) ~~\mapsto~~ \Upsilon({\rm e}^{{\rm i}\alpha}\zeta) \quad \Longleftrightarrow \quad \Upsilon_n(z) ~~\mapsto~~ {\rm e}^{{\rm i}n\alpha}\Upsilon_n(z)~. \tag{2.14}$$

The complex linear tangent variables Σ's in (2.13) can be dualized into chiral one-forms, in accordance with the generalized Legendre transform [6]. The target space for the model thus obtained is (an open domain of the zero section) of the cotangent bundle of the Kähler manifold $\mathcal{M}$ [10].

3. General construction

In what follows, we restrict our consideration to the case when $\mathcal{M}$ is a Hermitian symmetric space, hence

$$\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = 0 \ . \tag{3.1}$$

Then, the algebraic equations of motion (2.10) are known to be equivalent to the holomorphic geodesic equation (with complex evolution parameter) [10, 11]

$$\frac{\mathrm{d}^2 \Upsilon^I_*(\zeta)}{\mathrm{d}\zeta^2} + \Gamma^I{}_{JK}\Big(\Upsilon_*(\zeta), \bar{\Phi}\Big) \frac{\mathrm{d}\Upsilon^J_*(\zeta)}{\mathrm{d}\zeta} \frac{\mathrm{d}\Upsilon^K_*(\zeta)}{\mathrm{d}\zeta} = 0 \,, \tag{3.2}$$

under the same initial conditions (2.11). Here $\Gamma^I{}_{JK}(\Phi, \bar{\Phi})$ are the Christoffel symbols for the Kähler metric $g_{I\bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$. In particular, we have

$$\Upsilon^I_2 = -\frac{1}{2} \Gamma^I{}_{JK}\big(\Phi, \bar{\Phi}\big)\, \Sigma^J \Sigma^K \ . \tag{3.3}$$

According to the principles of projective superspace [6, 7], the action (2.1) is invariant under $\mathcal{N} = 2$ supersymmetry transformations

$$\delta \Upsilon(\zeta) = \mathrm{i}\left(\varepsilon^\alpha_i Q^i_\alpha + \bar{\varepsilon}^i_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}_i\right) \Upsilon(\zeta) \tag{3.4}$$

when $\Upsilon(\zeta)$ is viewed as a $\mathcal{N} = 2$ superfield. However, since the action is given in $\mathcal{N} = 1$ superspace, it is only the $\mathcal{N} = 1$ supersymmetry which is manifestly realized. The second hidden supersymmetry can be shown to act on the physical superfields Φ and Σ as follows (see, e.g., [8]):

$$\delta \Phi = \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Sigma \,, \qquad \delta \Sigma = -\varepsilon^\alpha D_\alpha \Phi + \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Upsilon_2 \ . \tag{3.5}$$

Upon elimination of the auxiliary superfields, the action (2.13), which is associated with the hermitian symmetric space $\mathcal{M}$, is invariant under

$$\delta \Phi^I = \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Sigma^I \,, \qquad \delta \Sigma^I = -\varepsilon^\alpha D_\alpha \Phi^I - \frac{1}{2} \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Big\{ \Gamma^I{}_{JK}\big(\Phi, \bar{\Phi}\big)\, \Sigma^J \Sigma^K \Big\} \ . \tag{3.6}$$

It turns out that the requirement of invariance under these transformations allows one to uniquely determine, by making use of (3.1), the tangent-bundle action (2.13). One finds

$$\begin{aligned} \mathcal{L}^{(1)} &= -g_{I\bar{J}}\big(\Phi, \bar{\Phi}\big) \Sigma^I \bar{\Sigma}^{\bar{J}} \,, \\ \mathcal{L}^{(n+1)} &\equiv \mathcal{L}_{I_1 \cdots I_{n+1} \bar{J}_1 \cdots \bar{J}_{n+1}} \Sigma^{I_1} \ldots \Sigma^{I_{n+1}} \bar{\Sigma}^{\bar{J}_1} \ldots \bar{\Sigma}^{\bar{J}_{n+1}} \\ &= -\frac{n}{2(n+1)}\, \mathcal{L}_{I_1 \cdots I_{n-1} L \bar{J}_1 \cdots \bar{J}_n}\, \Sigma^{I_{n+1}} \bar{\Sigma}^{\bar{J}_{n+1}}\, R_{I_{n+1} \bar{J}_{n+1} I_n}{}^L \Sigma^{I_1} \ldots \Sigma^{I_n} \bar{\Sigma}^{\bar{J}_1} \ldots \bar{\Sigma}^{\bar{J}_n} \ . \end{aligned} \tag{3.7}$$

It is useful to introduce (conjugate to each other) first-order differential operators

$$\begin{aligned} \mathcal{R}_{\Sigma, \bar{\Sigma}} &= -\frac{1}{2} \Sigma^K \bar{\Sigma}^{\bar{L}}\, R_{K\bar{L}I}{}^J\big(\Phi, \bar{\Phi}\big)\, \Sigma^I \frac{\partial}{\partial \Sigma^J} \,, \\ \bar{\mathcal{R}}_{\Sigma, \bar{\Sigma}} &= \frac{1}{2} \Sigma^K \bar{\Sigma}^{\bar{L}}\, R_{K\bar{L}\bar{I}}{}^{\bar{J}}\big(\Phi, \bar{\Phi}\big)\, \bar{\Sigma}^{\bar{I}} \frac{\partial}{\partial \bar{\Sigma}^{\bar{J}}} = -\frac{1}{2} \Sigma^K \bar{\Sigma}^{\bar{L}}\, R_{K\bar{L}}{}^{\bar{J}}{}_{\bar{I}}\big(\Phi, \bar{\Phi}\big)\, \bar{\Sigma}^{\bar{I}} \frac{\partial}{\partial \bar{\Sigma}^{\bar{J}}} \ . \end{aligned} \tag{3.8}$$

Since the metric and the curvature tensor are covariantly constant, we have

$$[\nabla_K, \bar{\nabla}_{\bar{L}}]\mathcal{L}_{I_1\cdots I_n \bar{J}_1\cdots \bar{J}_n} = 0\,, \tag{3.9}$$

and hence

$$\mathcal{R}_{\Sigma,\bar{\Sigma}}\,\mathcal{L}^{(n)} = \bar{\mathcal{R}}_{\Sigma,\bar{\Sigma}}\,\mathcal{L}^{(n)}\ . \tag{3.10}$$

Now, the second relation in (3.7) can be rewritten as follows:

$$\mathcal{L}^{(n+1)} = \frac{1}{n+1}\,\mathcal{R}_{\Sigma,\bar{\Sigma}}\,\mathcal{L}^{(n)}\ . \tag{3.11}$$

This leads to

$$\begin{aligned}\mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big) &= \sum_{n=1}^{\infty}\mathcal{L}_{I_1\cdots I_n \bar{J}_1\cdots \bar{J}_n}\big(\Phi,\bar{\Phi}\big)\Sigma^{I_1}\ldots\Sigma^{I_n}\bar{\Sigma}^{\bar{J}_1}\ldots\bar{\Sigma}^{\bar{J}_n} = \sum_{n=1}^{\infty}\mathcal{L}^{(n)}\\ &= -g_{I\bar{J}}\bar{\Sigma}^{\bar{J}}\,\frac{\mathrm{e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}}-1}{\mathcal{R}_{\Sigma,\bar{\Sigma}}}\,\Sigma^I\ .\end{aligned} \tag{3.12}$$

It is useful to rewrite this Lagrangian using an auxiliary variable t:

$$\mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big) = -\int_0^1 \mathrm{d}t\, g_{I\bar{J}}\,\bar{\Sigma}^{\bar{J}}\mathrm{e}^{t\mathcal{R}_{\Sigma,\bar{\Sigma}}}\Sigma^I\ . \tag{3.13}$$

The relations (3.7) can be shown to be equivalent to the first-order differential equation

$$\frac{1}{2}R_{K\bar{J}L}{}^{I}\,\frac{\partial\mathcal{L}}{\partial\Sigma^I}\,\Sigma^K\Sigma^L + \frac{\partial\mathcal{L}}{\partial\bar{\Sigma}^{\bar{J}}} + g_{I\bar{J}}\,\Sigma^I = 0 \tag{3.14}$$

which is obeyed by $\mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big)$ given in (3.12). Indeed, the action (2.13) varies under (3.6) as follows:

$$\begin{aligned}\delta S_{\rm tb}[\Phi,\Sigma] &= \int \mathrm{d}^8z\,\Big\{\frac{\partial\mathcal{L}}{\partial\Phi^I} - \frac{\partial\mathcal{L}}{\partial\Sigma^K}\,\Gamma^K_{\ IJ}\Sigma^J\Big\}\,\bar{\varepsilon}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\Sigma^I\\ &- \int \mathrm{d}^8z\,\Big\{\frac{1}{2}R_{K\bar{J}L}{}^{I}\,\frac{\partial\mathcal{L}}{\partial\Sigma^I}\,\Sigma^K\Sigma^L + \frac{\partial\mathcal{L}}{\partial\bar{\Sigma}^{\bar{J}}} + g_{I\bar{J}}\,\Sigma^I\Big\}\,\bar{\varepsilon}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{\Phi}^{\bar{J}} + \text{c.c.}\end{aligned} \tag{3.15}$$

Here the variation in the first line vanishes, since the curvature is covariantly constant.

To construct a dual formulation, consider the first-order action

$$S = \int \mathrm{d}^8z\,\Big\{\,K\big(\Phi,\bar{\Phi}\big) + \mathcal{L}\big(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}\big) + \Psi_I\,\Sigma^I + \bar{\Psi}_{\bar{I}}\bar{\Sigma}^{\bar{I}}\Big\}\ , \tag{3.16}$$

where the tangent vector Σ^I is now complex unconstrained, while the one-form Ψ is chiral, $\bar{D}_{\dot{\alpha}}\Psi_I = 0$. This action can be shown to be invariant under the following supersymmetry transformations:

$$\begin{aligned}\delta\Phi^I &= \frac{1}{2}\bar{D}^2\big\{\overline{\varepsilon\theta}\,\Sigma^I\big\}\,,\\ \delta\Sigma^I &= -\varepsilon^\alpha D_\alpha\Phi^I - \frac{1}{2}\bar{\varepsilon}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\Big\{\Gamma^I_{\ JK}\big(\Phi,\bar{\Phi}\big)\,\Sigma^J\Sigma^K\Big\} - \frac{1}{2}\overline{\varepsilon\theta}\,\Gamma^I_{\ JK}\big(\Phi,\bar{\Phi}\big)\,\Sigma^J\bar{D}^2\Sigma^K\,,\\ \delta\Psi_I &= -\frac{1}{2}\bar{D}^2\Big\{\overline{\varepsilon\theta}\,K_I\big(\Phi,\bar{\Phi}\big)\Big\} + \frac{1}{2}\bar{D}^2\Big\{\overline{\varepsilon\theta}\,\Gamma^K_{\ IJ}\big(\Phi,\bar{\Phi}\big)\,\Sigma^J\Big\}\,\Psi_K\ .\end{aligned} \tag{3.17}$$

Varying Σ's and their conjugates in (3.16) using (3.13) and properties of the curvatures of Hermitian symmetric spaces gives

$$\begin{aligned}\bar{\Psi}_{\bar{J}} &= g_{I\bar{J}}\, \mathrm{e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}} \Sigma^I\,, \\ \Psi_I &= g_{I\bar{J}}\, \mathrm{e}^{\bar{\mathcal{R}}_{\Sigma,\bar{\Sigma}}} \bar{\Sigma}^{\bar{J}}\,. \end{aligned} \tag{3.18}$$

Inverting these relations should lead to the cotangent-bundle action

$$S_{\mathrm{ctb}}[\Phi,\Psi] = \int \mathrm{d}^8 z\, \Big\{ K\big(\Phi,\bar{\Phi}\big) + \mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\Big\}\,, \tag{3.19}$$

where

$$\begin{aligned}\mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big) &= \sum_{n=1}^{\infty} \mathcal{H}^{I_1\cdots I_n \bar{J}_1\cdots\bar{J}_n}\big(\Phi,\bar{\Phi}\big)\Psi_{I_1}\dots\Psi_{I_n}\bar{\Psi}_{\bar{J}_1}\dots\bar{\Psi}_{\bar{J}_n}\,, \\ \mathcal{H}^{I\bar{J}}\big(\Phi,\bar{\Phi}\big) &= g^{I\bar{J}}\big(\Phi,\bar{\Phi}\big)\,. \end{aligned} \tag{3.20}$$

On general grounds, the cotangent-bundle action should be invariant under the supersymmetry transformations induced from (3.17)

$$\begin{aligned}\delta\Phi^I &= \frac{1}{2}\bar{D}^2\big\{\overline{\varepsilon\theta}\,\Sigma^I\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\big\}\,, \\ \delta\Psi_I &= -\frac{1}{2}\bar{D}^2\Big\{\overline{\varepsilon\theta}\,K_I\big(\Phi,\bar{\Phi}\big)\Big\} + \frac{1}{2}\bar{D}^2\Big\{\overline{\varepsilon\theta}\,\Gamma^K{}_{IJ}\big(\Phi,\bar{\Phi}\big)\,\Sigma^J\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\Big\}\,\Psi_K\,, \end{aligned} \tag{3.21}$$

with

$$\Sigma^I\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big) = \frac{\partial}{\partial\Psi_I}\,\mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)\,. \tag{3.22}$$

The requirement of invariance under such transformations can be shown to be equivalent to the following nonlinear equation on $\mathcal{H}$:

$$\Sigma^I\, g_{I\bar{J}} - \frac{1}{2}\,\Sigma^K\Sigma^L\,R_{K\bar{J}L}{}^I\,\Psi_I = \bar{\Psi}_{\bar{J}}\,. \tag{3.23}$$

This equation also follows directly from (3.14) using the definition of the Ψ's, or if one wants, as a consequence of the superspace Legendre transform. (It can be explicitly checked that the relation is satisfied for the expressions in (3.18), as it should).

The relation (3.23) allows us to uniquely reconstruct $\mathcal{H}\big(\Phi,\bar{\Phi},\Psi,\bar{\Psi}\big)$ formally defined in (3.20).

As a simple illustration of the formalism developed, in appendix A we re-derive the model on the tangent bundle of $\mathbb{C}P^n$. The actual power of our method is revealed in next section where it is applied to derive a $\mathcal{N}=2$ supersymmetric sigma model on the tangent bundle of $E_6/\,\mathrm{SO}(10)\times\mathrm{U}(1)$.

4. The hermitian symmetric space $E_6/\mathrm{SO}(10)\times\mathrm{U}(1)$

The Kähler potential for the hermitian symmetric space $E_6/\mathrm{SO}(10)\times\mathrm{U}(1)$ was computed by several groups [14 − 17] in different but equivalent forms. Here we will use the Kähler potential derived in ref. [17] with the aid of the techniques developed in [18]. In order to comply with the notation adopted in [17], we will use Greek letters to label indices, lower indices for base-space ($\Phi^I \to \Phi_\alpha$) and tangent ($\Sigma^I \to \Sigma_\alpha$) variables, while upper indices will be used for one-forms ($\Psi_I \to \Psi^\alpha$).

Locally, the symmetric space $E_6/\mathrm{SO}(10)\times\mathrm{U}(1)$ can be described by complex variables Φ_α transforming in the spinor representation **16** of SO(10) and their conjugates.

$$\Phi_\alpha\,, \qquad \bar\Phi^\alpha := (\Phi_\alpha)^*\,, \qquad \alpha = 1,\dots,16\,. \tag{4.1}$$

The Kähler potential is

$$K(\Phi,\bar\Phi) = \ln\left(1 + \bar\Phi^\alpha\Phi_\alpha + \frac{1}{8}(\bar\Phi^\alpha(\sigma_A)_{\alpha\beta}\bar\Phi^\beta)(\Phi_\gamma(\sigma^\dagger_A)^{\gamma\delta}\Phi_\delta)\right)\,, \quad A = 1,\dots 10 \tag{4.2}$$

where $(\sigma_A)_{\alpha\beta} = (\sigma_A)_{\beta\alpha}$ are the 16×16 sigma-matrices which generate, along with their Hermitian-conjugates, $(\sigma^\dagger_A)^{\alpha\beta}$, the ten-dimensional Dirac matrices in the Weyl representation. The sigma-matrices obey the anti-commutation relations

$$(\sigma_A\sigma^\dagger_B + \sigma_B\sigma^\dagger_A)_\alpha{}^\beta = 2\delta_{AB}\delta_\alpha{}^\beta\,. \tag{4.3}$$

The Kähler metric can be shown to be

$$\begin{aligned} g^\alpha{}_\beta = \frac{\partial^2 K}{\partial\Phi_\alpha\partial\bar\Phi^\beta} = \frac{1}{Z}\Bigg\{ & \delta_\alpha{}^\beta + \frac{1}{2}(\sigma_A)_{\alpha\gamma}\bar\Phi^\gamma(\sigma^\dagger_A)^{\beta\delta}\Phi_\delta \\ & + \frac{1}{Z}\Bigg(- \bar\Phi^\alpha\Phi_\beta - \frac{1}{4}\bar\Phi^\alpha(\sigma_A)_{\beta\gamma}\bar\Phi^\gamma(\Phi^{\mathrm T}\sigma^\dagger_A\Phi) - \frac{1}{4}(\sigma^\dagger_A)^{\alpha\delta}\Phi_\delta\Phi_\beta(\bar\Phi^{\mathrm T}\sigma_A\bar\Phi) \\ & - \frac{1}{16}(\sigma^\dagger_B)^{\alpha\delta}\Phi_\delta(\sigma_A)_{\beta\gamma}\bar\Phi^\gamma(\Phi^{\mathrm T}\sigma^\dagger_A\Phi)(\bar\Phi^{\mathrm T}\sigma_B\bar\Phi)\Bigg)\Bigg\}\,, \end{aligned} \tag{4.4}$$

where $Z = 1+\bar\Phi^{\mathrm T}\Phi+\frac{1}{8}(\Phi^{\mathrm T}\sigma^\dagger_A\Phi)(\bar\Phi^{\mathrm T}\sigma_A\bar\Phi)$. Here we have used the fact that σ_A is symmetric.

Let us calculate the Lagrangian (3.12) for the case under consideration. In our notation, the first-order differential operator defined in (3.12) is

$$\mathcal{R}_{\Sigma,\bar\Sigma} = -\frac{1}{2}\Sigma_\alpha\bar\Sigma^\beta\Sigma_\gamma R^\alpha{}_\beta{}^\gamma{}_\delta(g^{-1})^\delta{}_\epsilon\frac{\partial}{\partial\Sigma_\epsilon}\,. \tag{4.5}$$

where $(g^{-1})^\beta{}_\alpha = (g^\alpha{}_\beta)^{-1}$ is the inverse metric of $g^\alpha{}_\beta$, that is $g^\alpha{}_\gamma(g^{-1})^\gamma{}_\beta = \delta^\alpha{}_\beta$. Since we are considering a symmetric space, it is actually sufficient to carry out the calculations of our interest at a particular point, say at $\Phi = 0$. The Riemann tensor at $\Phi = 0$ can be shown to be

$$\begin{aligned} R^\alpha{}_\beta{}^\gamma{}_\delta\Big|_{\Phi=0} &= \partial^g\partial_\delta g^\alpha{}_\beta - (g^{-1})^\lambda{}_\kappa\partial^\kappa g^\alpha{}_\beta\partial_\lambda g^\gamma{}_\delta\Big|_{\Phi=0} \\ &= -\delta_\delta{}^\alpha\delta_\beta{}^\gamma + \frac{1}{2}(\sigma_A)_{\beta\delta}(\sigma^\dagger_A)^{\alpha\gamma} - \delta_\beta{}^\alpha\delta_\delta{}^\gamma\,. \end{aligned} \tag{4.6}$$

Now, simple calculations give

$$
\begin{aligned}
\mathcal{R}_{\Sigma,\bar{\Sigma}}\Sigma_\alpha &= |\Sigma|^2\Sigma_\alpha - \frac{1}{4}(\bar{\Sigma}^{\rm T}\sigma_A)_\alpha(\Sigma^{\rm T}\sigma_A^\dagger\Sigma)\,,\\
(\mathcal{R}_{\Sigma,\bar{\Sigma}})^2\Sigma_\alpha &= 2|\Sigma|^4\Sigma_\alpha - \frac{1}{2}(\bar{\Sigma}^{\rm T}\sigma_A)_\alpha|\Sigma|^2(\Sigma^{\rm T}\sigma_A^\dagger\Sigma) - \frac{1}{4}\Sigma_\alpha|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2\,,\\
(\mathcal{R}_{\Sigma,\bar{\Sigma}})^3\Sigma_\alpha &= 6|\Sigma|^6\Sigma_\alpha - \frac{3}{2}|\Sigma|^4(\bar{\Sigma}^{\rm T}\sigma_A)_\alpha(\Sigma^{\rm T}\sigma_A^\dagger\Sigma) - \frac{3}{2}\Sigma_\alpha|\Sigma|^2|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2\\
&\quad + \frac{3}{16}(\bar{\Sigma}^{\rm T}\sigma_A)_\alpha(\Sigma^{\rm T}\sigma_A^\dagger\Sigma)|\Sigma^{\rm T}\sigma_B^\dagger\Sigma|^2\,,
\end{aligned}
\tag{4.7}
$$

where $|\Sigma|^2 = \bar{\Sigma}^\alpha\Sigma_\alpha$ and $|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2 = \left(\Sigma^{\rm T}\sigma_A^\dagger\Sigma\right)\left(\bar{\Sigma}^{\rm T}\sigma_A\bar{\Sigma}\right)$. Here we have used the following identity

$$
\left(\sigma_A^\dagger\Phi\right)^\alpha\left(\Phi\sigma_A^\dagger\Phi\right) = 0 \tag{4.8}
$$

that follows from the Fierz identity

$$
\left(\epsilon\sigma_A^\dagger\psi\right)\left(\psi\sigma_A^\dagger\eta\right) = -\frac{1}{2}\left(\epsilon\sigma_A^\dagger\eta\right)\left(\psi\sigma_A^\dagger\psi\right)\ . \tag{4.9}
$$

Making use of the above results gives

$$
\begin{aligned}
\mathcal{L}(\Phi=0,\bar{\Phi}=0,\Sigma,\bar{\Sigma}) &= -g^\alpha_{\ \beta}\bar{\Sigma}^\beta\,\frac{{\rm e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}}-1}{\mathcal{R}_{\Sigma,\bar{\Sigma}}}\,\Sigma_\alpha\Big|_{\Phi=\bar{\Phi}=0}\\
&= -|\Sigma|^2 - \frac{1}{2}|\Sigma|^4 + \frac{1}{8}|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2 - \frac{1}{3}|\Sigma|^6 + \frac{1}{8}|\Sigma|^2|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2\\
&\quad - \frac{1}{4}|\Sigma|^8 + \frac{1}{8}|\Sigma|^4|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2 - \frac{1}{128}|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2|\Sigma^{\rm T}\sigma_B^\dagger\Sigma|^2 + \cdots
\end{aligned}
\tag{4.10}
$$

Looking at the expression obtained it is tempting to conjecture

$$
\mathcal{L}(\Phi=0,\bar{\Phi}=0,\Sigma,\bar{\Sigma}) = \ln\left(1 - |\Sigma|^2 + \frac{1}{8}|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2\right)\ . \tag{4.11}
$$

The latter relation extends to an arbitrary point Φ of the base manifold by replacing

$$
|\Sigma|^2 \ \to\ g^\alpha_{\ \beta}\Sigma_\alpha\bar{\Sigma}^\beta\,,\quad \frac{1}{8}|\Sigma^{\rm T}\sigma_A^\dagger\Sigma|^2 \ \to\ \frac{1}{2}(g^\alpha_{\ \beta}\Sigma_\alpha\bar{\Sigma}^\beta)^2 + \frac{1}{4}R^\alpha_{\ \beta\ \delta}^{\ \ \gamma}\Sigma_\alpha\bar{\Sigma}^\beta\Sigma_\gamma\bar{\Sigma}^\delta\ . \tag{4.12}
$$

Then one gets

$$
\begin{aligned}
\mathcal{L}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}) &= -g^\alpha_{\ \beta}\bar{\Sigma}^\beta\frac{{\rm e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}}-1}{\mathcal{R}_{\Sigma,\bar{\Sigma}}}\,\Sigma_\alpha\\
&= \ln\left(1 - g^\alpha_{\ \beta}\Sigma_\alpha\bar{\Sigma}^\beta + \frac{1}{2}(g^\alpha_{\ \beta}\Sigma_\alpha\bar{\Sigma}^\beta)^2 + \frac{1}{4}R^\alpha_{\ \beta\ \delta}^{\ \ \gamma}\Sigma_\alpha\bar{\Sigma}^\beta\Sigma_\gamma\bar{\Sigma}^\delta\right)\ .
\end{aligned}
\tag{4.13}
$$

This is actually the correct result for $\mathcal{L}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})$. Indeed, one can check that the r.h.s. of (4.13) satisfies the master equation (3.14) which in the present case reads

$$
\frac{1}{2}R^\alpha_{\ \beta\ \delta}^{\ \ \gamma}(g^{-1})^\delta_{\ \epsilon}\frac{\partial\mathcal{L}}{\partial\Sigma_\epsilon}\Sigma_\alpha\Sigma_\gamma + \frac{\partial\mathcal{L}}{\partial\bar{\Sigma}^\beta} + g^\alpha_{\ \beta}\Sigma_\alpha = 0\ . \tag{4.14}
$$

In order to prove this claim, it is sufficient to restrict our consideration to $\Phi = 0$. For the first term in the l.h.s. of (4.14), one finds

$$\frac{1}{2}R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}(g^{-1})^{\delta}{}_{\epsilon}\frac{\partial \mathcal{L}}{\partial \Sigma_{\epsilon}}\Sigma_{\alpha}\Sigma_{\gamma}\Big|_{\Phi=0} = \frac{1}{Z}\left(2\Sigma_{\beta}|\Sigma|^2 - \frac{1}{2}(\sigma_A\bar{\Sigma})_{\beta}(\Sigma^{\mathrm{T}}\sigma_A^{\dagger}\Sigma) - \frac{1}{4}\Sigma_{\beta}|\Sigma^{\mathrm{T}}\sigma_A^{\dagger}\Sigma|^2\right), \quad (4.15)$$

and this contribution exactly cancels against the other terms in (4.14).

Let us dualize the tangent-bundle action. For this purpose we consider the following first-order action

$$S = \int \mathrm{d}^8 z \Big\{ K(\Phi, \bar{\Phi}) + \ln\left(1 - g^{\alpha}_{\beta}U_{\alpha}\bar{U}^{\beta} + \frac{1}{2}(g^{\alpha}_{\beta}U_{\alpha}\bar{U}^{\beta})^2 + \frac{1}{4}R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}U_{\alpha}\bar{U}^{\beta}U_{\gamma}\bar{U}^{\delta}\right) + U_{\alpha}\Psi^{\alpha} + \bar{U}^{\alpha}\bar{\Psi}_{\alpha}\Big\}, \quad (4.16)$$

where the tangent variables U_α are complex unconstrained superfields, and the one-forms Ψ^α are chiral superfields, $\bar{D}_{\dot{\alpha}}\Psi = 0$. The variables U's and $\bar{U}$'s can be eliminated with the aid of their algebraic equations of motion. This turns the superfield Lagrangian into the hyperkähler potential

$$H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) - \ln\left(\Lambda + \sqrt{\Lambda + (g^{-1})^{\alpha}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha}}\right) + \Lambda + \sqrt{\Lambda + (g^{-1})^{\alpha}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha}} - \frac{2((g^{-1})^{\alpha}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha})^2 + \tilde{R}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}\bar{\Psi}_{\alpha}\Psi^{\beta}\bar{\Psi}_{\gamma}\Psi^{\delta}}{\Lambda + \sqrt{\Lambda + (g^{-1})^{\alpha}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha}}}, \quad (4.17)$$

where $\tilde{R}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = (g^{-1})^{\alpha}{}_{\alpha'}(g^{-1})^{\beta'}{}_{\beta}(g^{-1})^{\gamma}{}_{\gamma'}(g^{-1})^{\delta'}{}_{\delta}R^{\alpha'}{}_{\beta'}{}^{\gamma'}{}_{\delta'}$, and

$$\Lambda = \frac{1}{2} + \sqrt{\frac{1}{4} + (g^{-1})^{\alpha}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha} + 2((g^{-1})^{\alpha}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha})^2 + \tilde{R}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}\bar{\Psi}_{\alpha}\Psi^{\beta}\bar{\Psi}_{\gamma}\Psi^{\delta}}. \quad (4.18)$$

The derivation of the above results is given in appendix B.

Similar to eq. (4.14) in the tangent-bundle formulation, one can check that the hyperkähler potential (4.17) satisfies the equation (3.23), which in the present case takes the form

$$\Sigma_{\alpha}g^{\alpha}_{\beta} - \frac{1}{2}\Sigma_{\alpha}\Sigma_{\gamma}R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}(g^{-1})^{\delta}{}_{\epsilon}\Psi^{\epsilon} = \bar{\Psi}_{\beta}. \quad (4.19)$$

To prove this, we again set $\Phi = 0$. Then, the l.h.s. in (4.19) becomes

$$\Sigma_{\beta} - \frac{1}{2}\left(-2(\Sigma_{\alpha}\Psi^{\alpha})\Sigma_{\beta} + \frac{1}{2}(\sigma_A\Psi)_{\beta}(\Sigma^{\mathrm{T}}\sigma_A^{\dagger}\Sigma)\right). \quad (4.20)$$

Making here use of (B.2), we can express Ψ in terms of Σ. Then we have

$$\Sigma_{\beta} - \frac{1}{2}\Sigma_{\alpha}\Sigma_{\gamma}R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}(g^{-1})^{\delta}{}_{\epsilon}\Psi^{\epsilon}\Big|_{\Phi=0} = \frac{1}{\Omega}\left(\Sigma_{\beta} - \frac{1}{4}(\sigma_A\bar{\Sigma})_{\beta}(\Sigma^{\mathrm{T}}\sigma_A^{\dagger}\Sigma)\right), \quad (4.21)$$

where Ω is given in (B.1). Because of (B.2), the expression obtained is exactly $\bar{\Psi}_\beta$ at $\Phi = 0$.

5. An alternative formulation

In this section we give a reformulation of the Lagrangian defined by (3.7) which more directly relates it to our previous results. The reformulation requires certain identities to be satisfied for products of curvatures; we have not been able to determine if these identities are for a general Hermitean symmetric space. We define the operator $\mathbb{R}$ by

$$\mathbb{R} := \frac{1}{2}\Sigma^a\bar{\Sigma}^{\bar{b}}R_{a\bar{b}c}{}^{d}M_d^c \tag{5.1}$$

where M is the generator of the relevant structure group and acts on Σ as a transformation of a vector: $[X_b^a M_a^b, \Sigma^c] = X_b^c\Sigma^b$. Here a and $\bar{a}$ are tangent space indices. Using this we may in certain cases re-write the Lagrangian (3.12) as

$$\mathcal{L}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}) = -\eta_{a\bar{b}}\bar{\Sigma}^{\bar{b}}\ln(1+\mathbb{R})\mathbb{R}^{-1}\Sigma^a \tag{5.2}$$

where $\eta_{a\bar{b}}$ is the tangent space metric. The inverse $\mathbb{R}^{-1}$ is formal at this stage, but in the concrete examples that we want to consider it is always possible to make sense of it. The structure (5.2) is possible when the curvature satisfies

$$R_{N\bar{J}_1M\bar{J}_2}R_{I_1\bar{J}_3I_2}{}^{N}R_{I_4\bar{J}_4I_3}{}^{M} \propto R_{N\bar{J}_1I_1\bar{J}_2}R_{I_2\bar{J}_3M}{}^{N}R_{I_4\bar{J}_4I_3}{}^{M} \tag{5.3}$$

when symmetrized in $I_1\dots I_4$ and in $\bar{J}_1\dots\bar{J}_4$, and similar relations for higher products of curvatures. This is indeed true for the case of $\mathbb{C}P^n$ discussed in appendix A. We find that, at the origin,

$$\mathbb{R}\mathbb{R}^{-1}\Sigma^a = \Sigma^a \tag{5.4}$$

if we take

$$\mathbb{R}^{-1} = -\frac{r^2}{\Sigma\bar{\Sigma}}\delta_b{}^{c}M_c{}^{b} \tag{5.5}$$

which inserted in (5.2) leads to the Lagrangian

$$\mathcal{L}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}) = -\frac{r^2}{\Sigma\bar{\Sigma}}\bar{\Sigma}_a\ln(1+\mathbb{R})\Sigma^a \tag{5.6}$$

where all contractions and lowering of indices is done using $\eta_{a\bar{b}} = \delta_{ab}$ and we have

$$R_{a\bar{b}c\bar{d}} = -\frac{1}{r^2}\left(\delta_{ab}\delta_{cd} + \delta_{ad}\delta_{bc}\right)\ , \tag{5.7}$$

all evaluated at the origin (see appendix A for more details). Evaluating the expression (5.6) and re-expressing the result at an arbitrary point, we recover the standard form of the Lagrangian; (A.6).

Another case where the appropriate identities are satisfied is for the $\mathrm{SO}(n+2)/\,\mathrm{SO}(n)\times\mathrm{SO}(2)$-model discussed in section 6 in [9]. Here the metric at a point is as in the previous example, the curvature tensor at the origin is

$$R_{a\bar{b}c\bar{d}} = 2\left(-\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right)\ , \quad a = 1,\dots,n\ . \tag{5.8}$$

We may take

$$\mathbb{R}^{-1} = -\frac{1}{\Sigma^2\bar{\Sigma}^2}\Sigma_b\bar{\Sigma}^c M_c{}^b \tag{5.9}$$

to yield the following form of the Lagrangian

$$\mathcal{L}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}) = \frac{1}{\bar{\Sigma}^2}\bar{\Sigma}_a \ln(1+\mathbb{R})\bar{\Sigma}^a \ . \tag{5.10}$$

Evaluating the expression (5.10) and re-expressing the result at an arbitrary point, we recover the standard form of the Lagrangian [9].

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A. Example: complex projective space

As a simple example, consider the complex projective space $\mathbb{C}P^n = \mathrm{SU}(n+1)/\,\mathrm{U}(n)$ for which we have

$$K(\Phi,\bar{\Phi}) = r^2 \ln\left(1+\frac{1}{r^2}\Phi^L\overline{\Phi^L}\right) , \quad g_{I\bar{J}}(\Phi,\bar{\Phi}) = \frac{r^2\delta_{IJ}}{r^2+\Phi^L\overline{\Phi^L}} - \frac{r^2\overline{\Phi^I}\Phi^J}{(r^2+\Phi^L\overline{\Phi^L})^2} , \tag{A.1}$$

where $I,\bar{J} = 1,\dots,n$. It is sufficient to compute the Riemann curvature at $\Phi = 0$

$$R_{I_1\bar{J}_1I_2\bar{J}_2}\Big|_{\Phi=0} = K_{I_1\bar{J}_1I_2\bar{J}_2}\Big|_{\Phi=0} = -\frac{1}{r^2}\Big\{\delta_{I_1J_1}\delta_{I_2J_2}+\delta_{I_1J_2}\delta_{I_2J_1}\Big\} , \tag{A.2}$$

with all results below corresponding to the choice $\Phi = 0$. One gets

$$\Sigma^{I_1}\bar{\Sigma}^{\bar{J}_1}\Sigma^{I_2}\, R_{I_1\bar{J}_1I_2\bar{J}_2} = -\frac{2}{r^2}|\Sigma|^2\Sigma^{J_2} , \qquad |\Sigma|^2 = \delta_{IJ}\Sigma^I\bar{\Sigma}^{\bar{J}} , \tag{A.3}$$

and hence

$$\mathcal{R}_{\Sigma,\bar{\Sigma}} = \frac{1}{r^2}|\Sigma|^2\,\Sigma^L\frac{\partial}{\partial\Sigma^L} \ . \tag{A.4}$$

From here

$$\big(\mathcal{R}_{\Sigma,\bar{\Sigma}}\big)^n\Sigma^I = n!\,\frac{|\Sigma|^{2n}}{r^{2n}}\,\Sigma^I \tag{A.5}$$

and hence

$$-g_{I\bar{J}}\bar{\Sigma}^{\bar{J}}\,\frac{\mathrm{e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}}-1}{\mathcal{R}_{\Sigma,\bar{\Sigma}}}\,\Sigma^I = r^2\ln\left(1-\frac{1}{r^2}\,g_{I\bar{J}}(\Phi,\bar{\Phi})\,\Sigma^I\bar{\Sigma}^{\bar{J}}\right) \ . \tag{A.6}$$

This agrees with the previous calculations [11, 19].

B. Derivation of (4.17)

This appendix is devoted to the derivation of the hyperkähler potential (4.17). Since the base manifold is symmetric space, it is sufficient to perform the dualization, for the action (4.16), at $\Phi = 0$. Then, the first order Lagrangian

$$\mathcal{L} = \ln \Omega + U_\alpha \psi^\alpha + \bar{U}^\alpha \bar{\psi}_\alpha \,, \qquad \Omega = 1 - \bar{U}^{\mathrm{T}} U + \frac{1}{8} |U^{\mathrm{T}} \sigma_A U|^2 \,, \tag{B.1}$$

leads to the following equations of motion for $\bar{U}$'s and U's:

$$\frac{-U_\alpha + (\sigma_A \bar{U})_\alpha (U^{\mathrm{T}} \sigma_A^\dagger U)/4}{\Omega} = \bar{\psi}_\alpha \,, \quad \frac{-\bar{U}^\alpha + (\sigma_A^\dagger U)^\alpha (\bar{U}^{\mathrm{T}} \sigma_A \bar{U})/4}{\Omega} = \psi^\alpha \,, \tag{B.2}$$

where ψ is a cotangent vector at $\Phi = 0$ (it is useful to reserve the notation Ψ for a one-form at a generic point Φ of the base manifold). These equations imply

$$\bar{\psi}^{\mathrm{T}} \sigma_A^\dagger \bar{\psi} = \frac{U^{\mathrm{T}} \sigma_A^\dagger U}{\Omega} \,, \qquad \psi^{\mathrm{T}} \sigma_A \psi = \frac{\bar{U}^{\mathrm{T}} \sigma_A \bar{U}}{\Omega} \,, \tag{B.3}$$

and also

$$\frac{1}{4} + \bar{\psi}^{\mathrm{T}} \psi + \frac{1}{2} |\psi^{\mathrm{T}} \sigma_A \psi|^2 = \left(\frac{1}{2} + \frac{\bar{U}^{\mathrm{T}} U}{\Omega} \right)^2 \,. \tag{B.4}$$

By construction, the correspondence between the tangent and cotangent variables should be such that $U \to 0 \Leftrightarrow \psi \to 0$. This means that we have to choose the "plus" solution of (B.4), that is

$$\frac{\bar{U}^{\mathrm{T}} U}{\Omega} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \bar{\psi}^{\mathrm{T}} \psi + \frac{1}{2} |\psi^{\mathrm{T}} \sigma_A \psi|^2} \,. \tag{B.5}$$

Now, the results obtained above can be used to express Ω via ψ and its conjugate. By definition, we have

$$\frac{1}{\Omega} = \frac{1}{\Omega^2} - \frac{\bar{U}^{\mathrm{T}} U}{\Omega^2} + \frac{1}{8} \left| \frac{\psi^{\mathrm{T}} \sigma_A \psi}{\Omega} \right|^2 \,, \tag{B.6}$$

This is equivalent to

$$\left(\frac{1}{\Omega} - \frac{\Lambda}{2} \right)^2 = \frac{\Lambda^2}{4} - \frac{1}{8} |\psi^{\mathrm{T}} \sigma_A \psi|^2 \,, \tag{B.7}$$

where

$$\Lambda = \frac{1}{2} + \sqrt{\frac{1}{4} + \bar{\psi}^{\mathrm{T}} \psi + \frac{1}{2} |\psi^{\mathrm{T}} \sigma_A \psi|^2} \,. \tag{B.8}$$

Since for $\psi \to 0$ we should have $\Omega \to 1$, it is necessary to choose the "plus" solution of (B.7), that is

$$\frac{1}{\Omega} = \frac{\Lambda}{2} + \sqrt{\frac{\Lambda^2}{4} - \frac{1}{8} |\psi^{\mathrm{T}} \sigma_A \psi|^2} = \frac{\Lambda}{2} + \frac{1}{2} \sqrt{\Lambda + \bar{\psi}^{\mathrm{T}} \psi} \,. \tag{B.9}$$

The above consideration corresponds to the origin, $\Phi = 0$, of the base manifold. To extend these results to an arbitrary point Φ of the base manifold, we should replace

$$\begin{aligned} \bar{\psi}^{\rm T}\psi &\rightarrow (g^{-1})^{\alpha}{}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha}\,, \\ \frac{1}{8}|\psi^{\rm T}\sigma_A\psi|^2 &\rightarrow \frac{1}{2}((g^{-1})^{\alpha}{}_{\beta}\Psi^{\beta}\bar{\Psi}_{\alpha})^2 + \frac{1}{4}\tilde{R}^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta}\bar{\Psi}_{\alpha}\Psi^{\beta}\bar{\Psi}_{\gamma}\Psi^{\delta}\,. \end{aligned} \tag{B.10}$$

As a result, we arrive at (4.17).

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